ELECTROMAGNETIC THEORY

Concept of Scalar Field

✓ A scalar field, means every point in a region of space corresponds to a scalar quantity.

Example

Consider temperature in a room in your house. Room is a region of space where a 'temperature' field exists. Normally, we consider only single value for temperature of a room where we consider the average temperature. This concept of room temperature vanishes when the room under consideration is a kitchen; the temperature would be higher when you are close to stove and would be lower elsewhere.

So temperature is a 'field' that can be associated with every point inside the room.

The temperature field T(x,y,z) is a scalar field because the field quantity "temperature" is a scalar.

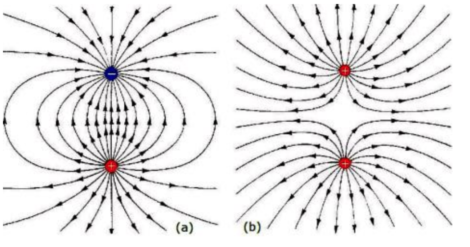
Concept of Vector Field

✓ A vector field, means every point in a region of space that corresponds to a vector quantity.

Examples:

1. Consider wind velocity inside a room. The contributing factors in this case are fans, open window, open doors ,etc. Wind velocity can be different (in magnitude and direction) in different points inside the room.

2. Consider the interaction between two charged particles. The lines of force shows the direction of 'electric filed'.



<u>Directional Derivative</u>

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

👍 It is a vector form of the partial differential operator and is called 'del'

Gradient { ∇A }

☞ Gradient of a scalar field is a vector field and its direction is normal to the level surface.

(1) Magnitude of the gradient at a point is the maximum possible magnitude of the directional derivative at that point, and

(2) Direction of the gradient is that direction in which the directional derivative takes maximum value.

Divergence

👍 The dot product of ∇ with a vector function is known as divergence of the vector function.

Let $\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z$

$$\boldsymbol{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\mathbf{div}\,\vec{\mathbf{A}} \;\text{ or }\; \boldsymbol{\nabla}.\vec{\mathbf{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Physical significance

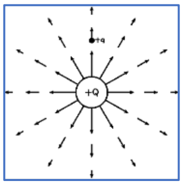
Divergence of a Vector filed is a Scalar and it is a measure of the amount of spread of the field at a point.

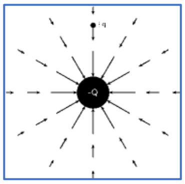
OR

Divergence of a vector function gives the net outflow (outflow minus inflow) per unit volume at a point.

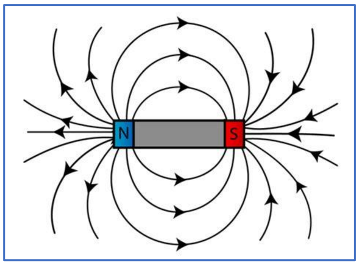
robink@vicet.org

If the outflow is greater than inflow ,divergence is positive and if the outflow is less than inflow ,divergence is negative.





In electrostatics, we see that the field produced by a positive charge has positive divergence while a negative charge produces an electrostatic field with negative divergence.



In case of a magnetic field, sine magnetic monopoles do not exist, the magnetic flux entering a unit volume is equal to that leaving the volume.

$$\nabla.\mathbf{B} = \mathbf{0}$$

Magnetic lines of force always form closed loops.

Curl

The cross product of **∇** with a vector function is known as curl of the vector function.

Consider a vector $\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z$

$$\text{curl}\,\vec{A} \;\text{ or }\; \nabla x\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla x\vec{A} = \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

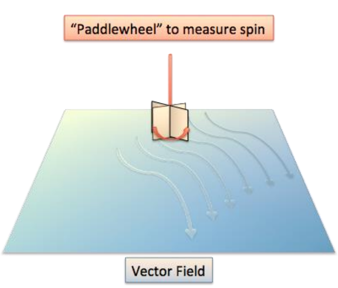
The name curl comes from “circulation” which measures how much does a vector field “curls” about a point.

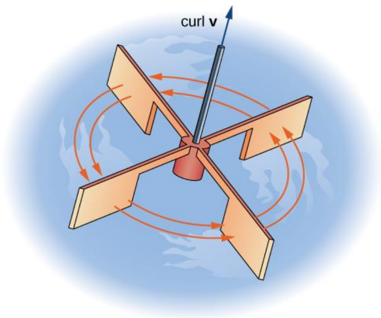
OR

Curl of a vector field measures the tendency of the vector field to rotate about a point.

✓ Curl of a vector field at a point is a vector that points in the direction of axis of rotation and has magnitude which represents the speed of rotation.

Consider flow of water and we want to determine if it has curl or not: is there any twisting or pushing force? To test this, we put a paddle wheel into the water and notice if it turns.
If the paddle does turn, it means this field has curl at that point. If it doesn't turn, then there's no curl.
What does it really mean if the paddle turns? Well, it means the water is pushing harder on one side than the other, making it twist. The larger the difference, the more forceful the twist and the bigger the curl. Also, a turning paddle wheel indicates that the field is "uneven" and not symmetric; if the field were even, then it would push on all sides equally and the paddle wouldn't turn at all.





Line Integral, Surface Integral & Volume Integral

Lines integral means the path is under consideration is to be divided into infinitesimal segments and the quantity in question is to be evaluated for each segments and the sum is taken for the entire path.

Surface integral means the surface is under consideration is to be divided into infinitesimal elemental areas and the quantity in question is to be evaluated for each elemental area and the sum is taken for the entire surface.

Volume integral means the volume is under consideration is to be divided into infinitesimal elemental volumes and the quantity in question is to be evaluated for each elemental volume and the sum is taken for the entire volume.

Stokes Theorem

The surface integral of the curl of a vector function taken over a surface is equal to the line integral of the vector function taken over the boundary of the surface.

$$\int_s (\nabla \mathrm{x} \vec{A}).\vec{ds} = \oint_l \vec{A}.\vec{dl}$$

Gauss's Divergence Theorem

The volume integral of divergence of a vector function is equal to the surface integral of the vector function taken over a closed surface.

$$\int_V (\nabla.\vec{A})dv = \int_S \vec{A}.\vec{ds}$$

Equation of Continuity

The total current flowing out of some volume must be equal to the rate of decrease of change within the volume. This indicates the conservation of charge.

$$\int_s \vec{J}.\vec{ds} = -\frac{\partial}{\partial t}\int_v \rho \, dv$$

Where ρ is the charge per unit volume. If the region of integration is stationary,

$$\int_s \vec{J}.\vec{ds} = -\int_v \frac{\partial \rho}{\partial t} dv$$

By using Gauss theorem,

$$\int_s \vec{J}.\vec{ds} = \int_V (\nabla.\vec{J})dv$$

$$\int_V (\nabla.\vec{J})dv = -\int_v \frac{\partial \rho}{\partial t} dv$$

$$\nabla.\vec{J} = \frac{-\partial \rho}{\partial t}$$

$$\boxed{\nabla.\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This is the equation of continuity, Where ρ is charge per unit volume or charge density and J is the current density

Maxwell's Equations

The electromagnetic wave phenomena are governed by a set of four equations known as Maxwell's field equations. Through these equations , Maxwell unified the laws of electricity and magnetism

$$\nabla.\vec{D} = \rho$$
$$\nabla.\vec{B} = 0$$
$$\nabla x\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla x\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

First equation is the statement of Gauss's Law in electrostatics. It is also called electric flux equation. Second equation explains Gauss's Law in magnetism. Third equation describe Faraday's law of electromagnetic induction. Fourth equation is the statement of modified Ampere's circuital law

Derivation of First Equation

According to Gauss' law of electrostatics, the total normal electric flux through a closed surface is equal to $1/\varepsilon_o$ times the charge enclosed in the volume

$$\int_s \vec{E}.\vec{ds} = \frac{1}{\varepsilon_0}\int_v \rho\, dv$$

$$\int_s \varepsilon_0\vec{E}.\vec{ds} = \int_v \rho\, dv$$

$$\int_s \vec{D}.\vec{ds} = \int_v \rho\, dv$$

Where ρ is charge density , E-electric field intensity and D –electric displacement vector

Using Gauss divergence theorem

$$\int_s \vec{D}.\vec{ds} = \int_v (\nabla.\vec{D})dv$$

$$\int_v (\nabla.\vec{D})dv = \int_v \rho\, dv$$

$$\nabla.\vec{D} = \rho$$ This is Maxwell's first equation

Derivation of Second Equation

By Gauss' law of magnetism, the net magnetic flux emerging through any closed surface is zero

$$\int_s \vec{B}.\vec{ds} = 0$$

By using Gauss divergence theorem,

$$\int_s \vec{B}.\vec{ds} = \int_v (\nabla.\vec{B})dv$$

$$\int_v (\nabla.\vec{B})dv = 0$$

This equation holds good for any arbitrary volume V

$$\nabla.\vec{B} = 0$$ This is Maxwell's second equation

Derivation of Third Equation

According to Faraday's law of electromagnetic induction, induced emf in a circuit is proportional to negative times rate of change of magnetic flux.

$$V = -\frac{\partial \phi}{\partial t} \quad \text{------------------------------- (1)}$$

Where $\phi = \int_s \vec{B}.\vec{ds}$ ------------------------------- (2)

Emf is defined as work done in taking a unit positive charge around closed path.

$$V = \oint_l \vec{E}.\vec{dl} \quad \text{------------------------------- (3)}$$

Using equation (2) and (3) in eqn (1)

$$\oint_l \vec{E}.\vec{dl} = -\int_s \frac{\partial \vec{B}}{\partial t}.\vec{ds} \quad \text{------------------------------- (4)}$$

By using Stoke's theorem

$$\oint_l \vec{E}.\vec{dl} = \int_s (\nabla x\vec{E}).\vec{ds} \quad \text{------------------------------- (5)}$$

From eqns (4) and (5)

$$\int_s (\nabla x\vec{E}).\vec{ds} = -\int_s \frac{\partial \vec{B}}{\partial t}.\vec{ds}$$

$$\nabla x\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ This is Maxwell's third equation

Derivation of Fourth Equation

Ampere's circuital law states that line integral of magnetic flux density B around a closed path is equal to μ_0 times the total current enclosed in the path

$$\oint_l \vec{B}.\vec{dl} = \mu_0 \mathbf{I}$$

Since $\vec{B} = \mu_0\vec{H}$ and $\mathbf{I} = \int_s \vec{J}.\vec{ds}$

$$\oint_l \mu_0\vec{H}.\vec{dl} = \mu_0 \int_s \vec{J}.\vec{ds}$$

$$\oint_l \vec{H}.\vec{dl} = \int_s \vec{J}.\vec{ds}$$

By using Stoke's theorem on LHS

$$\int_s (\nabla x\vec{H}).\vec{ds} = \int_s \vec{J}.\vec{ds}$$

$$\nabla x\vec{H} = \vec{J} \quad \text{------------------------------- (1)}$$

Maxwell showed that ampere's circuital law is inconsistent . Taking divergence on both sides of eqn (1)

$$\nabla.\nabla x\vec{H} = \nabla.\vec{J} \quad \text{------------------------------- (2)}$$

Divergence of curl of a function is zero which means.

$$\nabla.\vec{J} = 0 \quad \text{------------------------------- (3)}$$

This violates the general equation of continuity

$$\nabla.\vec{J} = -\frac{\partial\rho}{\partial t} \qquad \text{-- (4)}$$

So Maxwell removed the inconsistency by introducing the concept of displacement current. Using the Maxwell's first law

$$\nabla.\vec{J} = -\frac{\partial(\nabla.\vec{D})}{\partial t}$$

$$\nabla.[\vec{J} + \frac{\partial\vec{D}}{\partial t}] = 0 \qquad \text{-- (5)}$$

Equation (5) replaces eqn(3) in the general case.

The total current density consists of two terms ; conduction current density $\vec{J}$ and displacement current density $\frac{\partial\vec{D}}{\partial t}$. Maxwell modified eqn (1) as

$$\nabla x\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$ This is Maxwell's fourth equation

Conduction Current

Conduction Current is due to movement of electric charges in a conductor when an electric field is applied. Consider a conductor of length L and cross sectional area A. Current **I** flows on applying a potential difference **V** (electric field intensity **E**).

$$V = EL$$

Resistance of the conductor is

$$R = \frac{L}{\sigma A}$$ σ is conductivity

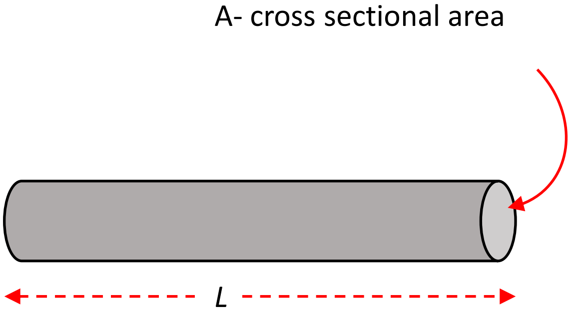
By Ohm's law $V = IR$

$$EL = IR$$

$$EL = I\frac{L}{\sigma A}$$

$$\sigma E = \frac{I}{A}$$

$$\vec{J} = \sigma\vec{E}$$



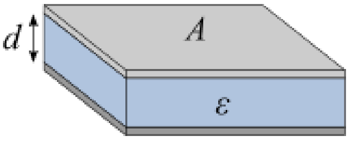
J is the conduction current per unit area

Displacement Current

Displacement Current is due to electric field that changes with time.

Consider a parallel plate capacitor of capacitance C. If the capacitor is charging and discharging, then

$$I = C\frac{dV}{dt} \qquad C = \frac{\varepsilon A}{d} \quad \text{and} \quad V = Ed$$



Current density $J = \frac{I}{A}$

$$J = \frac{C}{A}\frac{dV}{dt}$$

$$J = \frac{\varepsilon A}{Ad}\frac{d(Ed)}{dt} = \frac{\varepsilon Ad}{Ad}\frac{dE}{dt}$$

$$J = \frac{\varepsilon dE}{dt}$$

$$\vec{J} = \frac{d\vec{D}}{dt}$$

This is displacement current which arises only when there is a change in electric field

Conduction Current	**Displacement Current**
It is due to flow of electric charge	It is due to electric field that change with time
It obey Ohm's law	It does not obey Ohm's law
Conduction current density is represented by $J = \sigma \mathrm{E}$	Displacement current density is represented by $J = \frac{\varepsilon dE}{dt}$
It is the actual current	It is apparent current produced by time varying electric field
Conduction current in perfect vacuum is zero.	Displacement current has finite value in vacuum.

Electromagnetic Waves in Free Space

Maxwell's equations are

$$\nabla.\vec{D} = \rho$$

$$\nabla.\vec{B} = 0$$

$$\nabla x\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla x\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

In free space σ=0 and ρ=0 and therefore Maxwell's equations for free space are

$$\nabla.\vec{E} = 0 \quad \text{---(1)}$$

$$\nabla.\vec{H} = 0 \quad \text{---(2)}$$

$$\nabla x\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t} \quad \text{---(3)}$$

$$\nabla x\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad \text{---(4)}$$

Taking curl on both sides of eqn(3)

$$\nabla x(\nabla x\vec{E}) = -\mu_0\frac{\partial(\nabla x\vec{H})}{\partial t}$$

$$\nabla x\left(\nabla x\vec{E}\right) = \nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E} \qquad \nabla x\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}(\varepsilon_0\frac{\partial\vec{E}}{\partial t})$$

$$\nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

From eqn (1) $\nabla.\vec{E} = 0$

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{---(4)}$$

Taking curl on both sides of eqn(4)

$$\nabla x(\nabla x\vec{H}) = \varepsilon_0 \frac{\partial(\nabla x\vec{E})}{\partial t}$$

By vector calculus $\nabla x(\nabla x\vec{H}) = \nabla(\nabla.\vec{H}) - \nabla^2\vec{H}$ and from eqn (3) $\nabla x\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$$\nabla(\nabla.\vec{H}) - \nabla^2\vec{H} = -\varepsilon_0 \frac{\partial}{\partial t}(\mu_0 \frac{\partial \vec{H}}{\partial t})$$

$$\nabla(\nabla.\vec{H}) - \nabla^2\vec{H} = -\mu_0\varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2}$$

From eqn (2) $\nabla.\vec{H} = 0$

$$\nabla^2\vec{H} = \mu_0\varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \qquad \text{---(5)}$$

Comparing equations (4) and (5) with standard wave equation shows that **E** and **H** are propagating as a waves

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

$$\frac{1}{v^2} = \mu_0\varepsilon_0$$

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Substituting the values μ_0 = 4π x 10^{-7} Hm^{-1} and ε_0 =8.85 x 10^{-12} Fm^{-1}

$$v = \frac{1}{\sqrt{8.85 \times 10^{-12}\ Fm^{-1} \times 4\pi \times 10^{-7}\ Hm^{-1}}} = 2.99794\times10^{8}\ ms^{-1}$$

$F = s^2H^{-1}$

This is same as the experimentally determined value of velocity of the light. This coincidence led Maxwell to assume that light is an electromagnetic wave.

Poynting's Theorem

When electromagnetic wave is propagated through space, energy is transferred from source to receiver. Poynting's theorem states that 'the rate of energy flow outward through unit area of a source in a direction normal to the surface is given by

$$\vec{S} = \vec{E} \times \vec{H}$$

Where $\vec{S}$ is the Poynting vector which represents the instantaneous power density vector associated with electromagnetic field at any point. Poynting vector $\vec{S}$ is normal to both $\vec{E}$ and $\vec{H}$.